

# A CHARACTERIZATION OF TWO WEIGHT TRACE INEQUALITIES FOR POSITIVE DYADIC OPERATORS IN THE UPPER TRIANGLE CASE

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ABSTRACT. Two weight trace inequalities for positive dyadic operators are characterized in terms of discrete Wolff's potentials in the upper triangle case  $1 < q < p < \infty$ .

## 1. INTRODUCTION

The purpose of this paper is to establish the two weight  $T1$  theorem for positive dyadic operators in the upper triangle case  $1 < q < p < \infty$ . We first fix some notations. We will denote  $\mathcal{D}$  by the family of all dyadic cubes  $Q = 2^{-i}(k + [0, 1)^n)$ ,  $i \in \mathbb{Z}$ ,  $k \in \mathbb{Z}^n$ . Let  $\sigma$  and  $\omega$  be nonnegative Radon measures on  $\mathbb{R}^n$  and let  $K : \mathcal{D} \rightarrow [0, \infty)$  be a map. For an  $f \in L^1_{\text{loc}}(d\sigma)$  the positive dyadic operator  $T_K[f d\sigma]$  is defined by

$$T_K[f d\sigma](x) := \sum_{Q \in \mathcal{D}} K(Q) \int_Q f d\sigma 1_Q(x) \quad x \in \mathbb{R}^n.$$

We will denote by  $\overline{K}_\sigma(Q)(x)$  the function

$$\overline{K}_\sigma(Q)(x) := \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} K(Q') \sigma(Q') 1_{Q'}(x), \quad x \in Q \in \mathcal{D},$$

and  $\overline{K}_\sigma(Q)(x) = 0$  when  $\sigma(Q) = 0$ . For  $s > 1$  discrete Wolff's potential of  $\omega$   $\mathcal{W}_{K,\sigma}^s[\omega](x)$  is defined by

$$\mathcal{W}_{K,\sigma}^s[\omega](x) := \sum_{Q \in \mathcal{D}} K(Q) \sigma(Q) \left( \int_Q \overline{K}_\sigma(Q)(y) d\omega(y) \right)^{s-1} 1_Q(x), \quad x \in \mathbb{R}^n.$$

The pair  $(K, \sigma)$  is said to satisfy the dyadic logarithmic bounded oscillation (DLBO) condition, if they fulfill

$$\sup_{x \in Q} \overline{K}_\sigma(Q)(x) \leq A \inf_{x \in Q} \overline{K}_\sigma(Q)(x),$$

where the constant  $A$  does not depend on  $Q \in \mathcal{D}$ . For each  $1 < p < \infty$ ,  $p'$  will denote the dual exponent of  $p$ , i.e.,  $p' = \frac{p}{p-1}$ .

In their significant paper [2], Cascante, Ortega and Verbitsky established the following:

**Proposition 1.1** ([2, Theorem A]). *Let  $0 < q < p < \infty$  and  $1 < p < \infty$ . Suppose that the pair  $(K, \sigma)$  satisfy the DLBO condition. Then two weight trace inequality*

$$(1.1) \quad \|T_K[f d\sigma]\|_{L^q(d\omega)} \leq C_1 \|f\|_{L^p(d\sigma)}$$

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holds if and only if

$$\|\mathcal{W}_{K,\sigma}^{p'}[\omega]^{1/p'}\|_{L^r(d\omega)} \leq C_2 < \infty, \text{ where } \frac{1}{q} = \frac{1}{r} + \frac{1}{p}.$$

Moreover, the least possible  $C_1$  and  $C_2$  are equivalent.

In his elegant paper [10] Sergei Treil gives a simple proof of the following two weight  $T1$  theorem for positive dyadic operators in the lower triangle case.

**Proposition 1.2** ([10, Theorem 2.1]). *Let  $1 < p \leq q < \infty$ . Then two weight trace inequality (1.1) holds if and only if*

$$\begin{cases} \sup_{Q \in \mathcal{D}} \frac{1}{\sigma(Q)^{1/p}} \left( \int_Q \left( \sum_{Q' \subset Q} K(Q') \omega(Q') 1_{Q'} \right)^q d\omega \right)^{1/q} \leq C_2 < \infty, \\ \sup_{Q \in \mathcal{D}} \frac{1}{\omega(Q)^{1/q'}} \left( \int_Q \left( \sum_{Q' \subset Q} K(Q') \sigma(Q') 1_{Q'} \right)^{p'} d\sigma \right)^{1/p'} \leq C_2 < \infty. \end{cases}$$

Moreover, the least possible  $C_1$  and  $C_2$  are equivalent.

Proposition 1.2 was first proved for  $p = 2$  in [6] by the Bellman function method. Later in [4] this was proved in full generality  $1 < p \leq q < \infty$ . The checking condition in Proposition 1.2 is called “Sawyer type checking condition”, since this was first introduced by Eric T. Sawyer in [7, 8].

In his excellent survey of the  $A_2$  theorem [3] Tuomas P. Hytönen introduces another proof of Proposition 1.2, which uses the “parallel corona” decomposition from the recent work of Lacey, Sawyer, Shen and Uriarte-Tuero [5] on the two weight boundedness of the Hilbert transform.

Following Hytönen’s arguments and applying a basic lemma due to [1], we shall establish the following two weight  $T1$  theorem for positive dyadic operators in the upper triangle case.

**Theorem 1.3.** *Let  $1 < q < p < \infty$ . Then two weight trace inequality (1.1) holds if and only if*

$$\begin{cases} \|\mathcal{W}_{K,\omega}^q[\sigma]^{1/q}\|_{L^r(d\sigma)} \leq C_2 < \infty, \\ \|\mathcal{W}_{K,\sigma}^{p'}[\omega]^{1/p'}\|_{L^r(d\omega)} \leq C_2 < \infty, \\ \text{where } \frac{1}{q} = \frac{1}{r} + \frac{1}{p}. \end{cases}$$

Moreover, the least possible  $C_1$  and  $C_2$  are equivalent.

**Remark 1.4.** The DLBO condition is essential and quite useful. In [9], we develop a theory of weights for positive operators in a filtered measure space based upon this condition.

The letter  $C$  will be used for constants that may change from one occurrence to another. Constants with subscripts, such as  $C_1, C_2$ , do not change in different occurrences.

## 2. PROOF OF THEOREM 1.3

In what follows we shall prove Theorem 1.3. We need a basic lemma [1, Theorem 2.1]. For the sake of completeness, we will give the proof and will also check the constants.

**Lemma 2.1.** *Let  $\sigma$  be a Radon measure on  $\mathbb{R}^n$ . Let  $1 < s < \infty$  and  $\{\alpha_Q\}_{Q \in \mathcal{D}} \subset [0, \infty)$ . Define, for  $Q_0 \in \mathcal{D}$ ,*

$$\begin{aligned} A_1 &:= \int_{Q_0} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^s d\sigma, \\ A_2 &:= \sum_{Q \subset Q_0} \alpha_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1}, \\ A_3 &:= \int_{Q_0} \sup_{x \in Q \subset Q_0} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^s d\sigma(x). \end{aligned}$$

Then

$$A_1 \leq c(s)A_2, \quad A_2 \leq c(s)^{\frac{1}{s-1}} A_3 \text{ and } A_3 \leq (s')^s A_1.$$

Here,

$$c(s) := \begin{cases} s, & 1 < s \leq 2, \\ (s(s-1) \cdots (s-k))^{\frac{s-1}{s-k-1}}, & 2 < s < \infty, \end{cases}$$

where  $k = \lceil s-2 \rceil$  is the smallest integer greater than  $s-2$ .

*Proof.* By a standard limiting argument, we may assume without loss of generality that there is only a finite number of  $\alpha_Q \neq 0$ .

(i) We prove  $A_1 \leq c(s)A_2$ . We use an elementary inequality

$$(2.1) \quad \left( \sum_i a_i \right)^s \leq s \sum_i a_i \left( \sum_{j \geq i} a_j \right)^{s-1},$$

where  $\{a_i\}_{i \in \mathbb{Z}}$  is a sequence of summable nonnegative reals. First, we verify the simple case  $1 < s \leq 2$ . It follows from (2.1) that

$$\begin{aligned} A_1 &= \int_{Q_0} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^s d\sigma \\ &\leq s \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} \int_Q \left( \sum_{Q' \subset Q} \frac{\alpha_{Q'}}{\sigma(Q')} 1_{Q'} \right)^{s-1} d\sigma \\ &\leq s \sum_{Q \subset Q_0} \alpha_Q \left( \frac{1}{\sigma(Q)} \int_Q \left( \sum_{Q' \subset Q} \frac{\alpha_{Q'}}{\sigma(Q')} 1_{Q'} \right)^{s-1} d\sigma \right)^{s-1} \\ &= s \sum_{Q \subset Q_0} \alpha_Q \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} = sA_2, \end{aligned}$$

where we have used  $s-1 \leq 1$  and Hölder's inequality. Next, we prove the case  $s > 2$ . Let  $k = \lceil s-2 \rceil$  be the smallest integer greater than  $s-2$ . Applying (2.1)  $(k+1)$ -times, we have

$$\begin{aligned} A_1 &= s(s-1) \cdots (s-k) \\ &\quad \times \sum_{P_k \subset \cdots \subset P_1 \subset P_0 \subset Q_0} \frac{\alpha_{P_0}}{\sigma(P_0)} \frac{\alpha_{P_1}}{\sigma(P_1)} \cdots \frac{\alpha_{P_k}}{\sigma(P_k)} \int_{P_k} \left( \sum_{P \subset P_k} \frac{\alpha_P}{\sigma(P)} 1_P \right)^{s-k-1} d\sigma. \end{aligned}$$

Since we have  $0 < s - k - 1 \leq 1$ ,

$$\begin{aligned} & \frac{1}{\sigma(P_k)} \int_{P_k} \left( \sum_{P \subset P_k} \frac{\alpha_P}{\sigma(P)} 1_P \right)^{s-k-1} d\sigma \\ & \leq \left( \frac{1}{\sigma(P_k)} \sum_{P \subset P_k} \alpha_P \right)^{s-k-1}. \end{aligned}$$

These yield

$$\begin{aligned} A_1 & \leq s(s-1) \cdots (s-k) \\ & \times \int_{Q_0} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^k \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-k-1} 1_Q \right) d\sigma. \end{aligned}$$

Hölder's inequality with exponent  $\frac{k}{s-1} + \frac{s-k-1}{s-1} = 1$  gives

$$\begin{aligned} & \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-k-1} 1_Q \\ & \leq \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^{\frac{k}{s-1}} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} 1_Q \right)^{\frac{s-k-1}{s-1}}, \end{aligned}$$

and, hence,

$$\begin{aligned} A_1 & \leq s(s-1) \cdots (s-k) \\ & \times \int_{Q_0} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right)^{\frac{ks}{s-1}} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} 1_Q \right)^{\frac{s-k-1}{s-1}} d\sigma. \end{aligned}$$

Hölder's inequality with the same exponent gives

$$A_1 \leq s(s-1) \cdots (s-k) A_1^{\frac{k}{s-1}} A_2^{\frac{s-k-1}{s-1}}.$$

Thus, we obtain  $A_1 \leq c(s)A_2$ .

(ii) We prove  $A_2 \leq c(s)^{\frac{1}{s-1}} A_3$ . It follows that

$$\begin{aligned} A_2 & = \int_{Q_0} \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} 1_Q d\sigma \\ & \leq \int_{Q_0} \left( \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q(x) \right) \left( \sup_{x \in Q \subset Q_0} \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^{s-1} d\sigma(x). \end{aligned}$$

Hölder's inequality gives

$$A_2 \leq A_1^{\frac{1}{s}} A_3^{\frac{1}{s'}}.$$

Since we have had  $A_1 \leq c(s)A_2$ , we obtain  $A_2 \leq c(s)^{\frac{1}{s-1}} A_3$ .

(iii) We prove  $A_3 \leq (s')^s A_1$ . It follows that

$$\begin{aligned} A_3 &= \int_{Q_0} \sup_{x \in Q \subset Q_0} \left( \frac{1}{\sigma(Q)} \sum_{Q' \subset Q} \alpha_{Q'} \right)^s d\sigma(x) \\ &\leq \int_{Q_0} M_\sigma \left[ \sum_{Q \subset Q_0} \frac{\alpha_Q}{\sigma(Q)} 1_Q \right] (x)^s d\sigma(x) \\ &\leq (s')^s A_1, \end{aligned}$$

where  $M_\sigma$  is the dyadic Hardy-Littlewood maximal operator and we have used the  $L^s(d\sigma)$ -boundedness of  $M_\sigma$ . This completes the proof.  $\square$

**Proof of Theorem 1.3 (Sufficiency):** We follow the arguments due to Hytönen in [3]. Let  $Q_0 \in \mathcal{D}$  be taken large enough and be fixed. We shall estimate the quantity

$$(2.2) \quad \sum_{Q \subset Q_0} K(Q) \int_Q f d\sigma \int_Q g d\omega,$$

where  $f \in L^p(d\sigma)$  and  $g \in L^{q'}(d\omega)$  are nonnegative and are supported in  $Q_0$ .

We define the collections of principal cubes  $\mathcal{F}$  for the pair  $(f, \sigma)$  and  $\mathcal{G}$  for the pair  $(g, \omega)$ . Namely, analogously for  $\mathcal{G}$ ,

$$\mathcal{F} := \bigcup_{k=0}^{\infty} \mathcal{F}_k,$$

where  $\mathcal{F}_0 := \{Q_0\}$ ,

$$\mathcal{F}_{k+1} := \bigcup_{F \in \mathcal{F}_k} ch_{\mathcal{F}}(F)$$

and  $ch_{\mathcal{F}}(F)$  is defined by the set of all maximal dyadic cubes  $Q \subset F$  such that

$$\frac{1}{\sigma(Q)} \int_Q f d\sigma > \frac{2}{\sigma(F)} \int_F f d\sigma.$$

Observe that

$$\begin{aligned} &\sum_{F' \in ch_{\mathcal{F}}(F)} \sigma(F') \\ &\leq \left( \frac{2}{\sigma(F)} \int_F f d\sigma \right)^{-1} \sum_{F' \in ch_{\mathcal{F}}(F)} \int_{F'} f d\sigma \\ &\leq \left( \frac{2}{\sigma(F)} \int_F f d\sigma \right)^{-1} \int_F f d\sigma = \frac{\sigma(F)}{2}, \end{aligned}$$

and, hence,

$$(2.3) \quad \sigma(E_{\mathcal{F}}(F)) := \sigma \left( F \setminus \bigcup_{F' \in ch_{\mathcal{F}}(F)} F' \right) \geq \frac{\sigma(F)}{2},$$

where the sets  $E_{\mathcal{F}}(F)$  are pairwise disjoint.

We further define the stopping parents, for  $Q \in \mathcal{D}$ ,

$$\begin{cases} \pi_{\mathcal{F}}(Q) := \min\{F \supset Q : F \in \mathcal{F}\}, \\ \pi_{\mathcal{G}}(Q) := \min\{G \supset Q : G \in \mathcal{G}\}, \\ \pi(Q) := (\pi_{\mathcal{F}}(Q), \pi_{\mathcal{G}}(Q)). \end{cases}$$

Then we can rewrite the series in (2.2) as follows:

$$\begin{aligned} \sum_{Q \subset Q_0} &= \sum_{\substack{F \in \mathcal{F}, \\ G \in \mathcal{G}}} \sum_{\substack{Q: \\ \pi(Q)=(F,G)}} \\ &\leq \sum_{F \in \mathcal{F}} \sum_{G \subset F} \sum_{\substack{Q: \\ \pi(Q)=(F,G)}} + \sum_{G \in \mathcal{G}} \sum_{F \subset G} \sum_{\substack{Q: \\ \pi(Q)=(F,G)}} , \end{aligned}$$

where we have used the fact that if  $P, Q \in \mathcal{D}$  then  $P \cap Q \in \{P, Q, \emptyset\}$ . Since the proof can be done in completely symmetric way, we shall concentrate ourselves on the first case only.

It follows that, for  $F \in \mathcal{F}$ ,

$$\begin{aligned} &\sum_{G \subset F} \sum_{\substack{Q: \\ \pi(Q)=(F,G)}} K(Q) \int_Q f \, d\sigma \int_Q g \, d\omega \\ &= \sum_{G \subset F} \sum_{\substack{Q: \\ \pi(Q)=(F,G)}} K(Q) \sigma(Q) \left( \frac{1}{\sigma(Q)} \int_Q f \, d\sigma \right) \int_Q g \, d\omega \\ &\leq \frac{2}{\sigma(F)} \int_F f \, d\sigma \sum_{G \subset F} \sum_{\substack{Q: \\ \pi(Q)=(F,G)}} K(Q) \sigma(Q) \int_Q g \, d\omega. \end{aligned}$$

We need the two observations. Suppose that  $\pi(Q) = (F, G)$  and  $G \subset F$ . If  $F' \in ch_{\mathcal{F}}(F)$  satisfies  $F' \subset Q$ , then by definition of  $\pi_{\mathcal{F}}$  we must have

$$\pi_{\mathcal{F}}(\pi_{\mathcal{G}}(F')) = F.$$

By this observation we define

$$ch_{\mathcal{F}}^*(F) := \{F' \in ch_{\mathcal{F}}(F) : \pi_{\mathcal{F}}(\pi_{\mathcal{G}}(F')) = F\}.$$

We further observe that, when  $F' \in ch_{\mathcal{F}}^*(F)$ , we can regard  $g$  as a constant on  $F'$  in the above integrals. By these observations we see that, by use of Hölder's inequality,

$$\begin{aligned} &\sum_{G \subset F} \sum_{\substack{Q: \\ \pi(Q)=(F,G)}} K(Q) \sigma(Q) \int_Q g \, d\omega \\ &\leq \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) 1_Q \right)^q d\omega \right)^{1/q} \\ &\quad \times \left( \int_{E_{\mathcal{F}}(F)} g^{q'} d\omega + \sum_{F' \in ch_{\mathcal{F}}^*(F)} \left( \frac{1}{\omega(F')} \int_{F'} g \, d\omega \right)^{q'} \omega(F') \right)^{1/q'} \\ &=: \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) 1_Q \right)^q d\omega \right)^{1/q} \|g_F\|_{L^{q'}(d\omega)}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
& \sum_{F \in \mathcal{F}} \sum_{G \subset F} \sum_{\substack{Q: \\ \pi(Q)=(F,G)}} K(Q) \int_Q f \, d\sigma \int_Q g \, d\omega \\
& \leq \sum_{F \in \mathcal{F}} \frac{2}{\sigma(F)} \int_F f \, d\sigma \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) 1_Q \right)^q d\omega \right)^{1/q} \|g_F\|_{L^{q'}(d\omega)} \\
& \leq 2 \left( \sum_{F \in \mathcal{F}} \left( \frac{1}{\sigma(F)} \int_F f \, d\sigma \right)^p \sigma(F) \right)^{1/p} \\
& \quad \times \left[ \sum_{F \in \mathcal{F}} \left\{ \frac{1}{\sigma(F)^{1/p}} \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) 1_Q \right)^q d\omega \right)^{1/q} \right\}^{p'} \|g_F\|_{L^{q'}(d\omega)}^{p'} \right]^{1/p'} \\
& =: 2I_1 \times I_2.
\end{aligned}$$

For  $I_1$ , using  $\sigma(F) \leq 2\sigma(E_{\mathcal{F}}(F))$ ,

$$\frac{1}{\sigma(F)} \int_F f \, d\sigma \leq \inf_{y \in F} M_{\sigma} f(y)$$

and the disjointness of the  $E_{\mathcal{F}}(F)$ , we have

$$\begin{aligned}
I_1 & \leq 2^{1/p} \left( \sum_{F \in \mathcal{F}} \int_{E_{\mathcal{F}}(F)} (M_{\sigma} f)^p \, d\sigma \right)^{1/p} \\
& \leq 2^{1/p} \left( \int_{Q_0} (M_{\sigma} f)^p \, d\sigma \right)^{1/p} \leq 2^{1/p} p' \|f\|_{L^p(d\sigma)}.
\end{aligned}$$

Recall that  $\frac{1}{q} = \frac{1}{r} + \frac{1}{p}$  and let  $\theta := \frac{q'}{p'}$ . Then we have  $\theta > 1$  and  $\theta' p' = r$ . It follows from Hölder's inequality with exponent  $\theta$  that

$$\begin{aligned}
I_2 & \leq \left[ \sum_{F \in \mathcal{F}} \left\{ \frac{1}{\sigma(F)^{1/p}} \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) 1_Q \right)^q d\omega \right)^{1/q} \right\}^r \right]^{1/r} \\
& \quad \times \left( \sum_{F \in \mathcal{F}} \|g_F\|_{L^{q'}(d\omega)}^{q'} \right)^{1/q'} \\
& =: I_{21} \times I_{22}.
\end{aligned}$$

It follows by applying Lemma 2.1 that

$$\begin{aligned}
& \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) 1_Q \right)^q d\omega \\
& \leq c(q) \sum_{Q \subset F} K(Q) \sigma(Q) \omega(Q) \left( \frac{1}{\omega(Q)} \sum_{Q' \subset Q} K(Q') \sigma(Q') \omega(Q') \right)^{q-1} \\
& = c(q) \int_F \sum_{Q \subset F} K(Q) \omega(Q) \left( \int_Q \bar{K}_{\omega}(Q)(y) \, d\sigma(y) \right)^{q-1} 1_Q \, d\sigma.
\end{aligned}$$

This implies

$$\begin{aligned}
& \left\{ \frac{1}{\sigma(F)^{1/p}} \left( \int_F \left( \sum_{Q \subset F} K(Q) \sigma(Q) 1_Q \right)^q d\omega \right)^{1/q} \right\}^r \\
& \leq c(q)^{r/q} \left( \frac{1}{\sigma(F)} \int_F \sum_{Q \subset F} K(Q) \omega(Q) \left( \int_Q \overline{K}_\omega(Q)(y) d\sigma(y) \right)^{q-1} 1_Q d\sigma \right)^{r/q} \sigma(F) \\
& \leq 2c(q)^{r/q} \int_{E_{\mathcal{F}}(F)} \left( M_\sigma \mathcal{W}_{K,\omega}^q[\sigma] \right)^{r/q} d\sigma,
\end{aligned}$$

and, hence,

$$I_{21} \leq 2^{1/q} c(q)^{1/q} (r/q)' \|\mathcal{W}_{K,\omega}^q[\sigma]^{1/q}\|_{L^r(d\sigma)}.$$

It remains to estimate  $I_{22}$ . it follows that

$$I_{22}^{q'} = \sum_{F \in \mathcal{F}} \int_{E_{\mathcal{F}}(F)} g^{q'} d\omega + \sum_{F \in \mathcal{F}} \sum_{F' \in ch_{\mathcal{F}}^*(F)} \left( \frac{1}{\omega(F')} \int_{F'} g d\omega \right)^{q'} \omega(F').$$

By the pairwise disjointness of the set  $E_{\mathcal{F}}(F)$ , it is immediate that

$$\sum_{F \in \mathcal{F}} \int_{E_{\mathcal{F}}(F)} g^{q'} d\omega \leq \|g\|_{L^{q'}(d\omega)}^{q'}.$$

For the remaining double sum, we use the definition of  $ch_{\mathcal{F}}^*(F)$  to reorganize:

$$\begin{aligned}
& \sum_{F \in \mathcal{F}} \sum_{F' \in ch_{\mathcal{F}}^*(F)} \left( \frac{1}{\omega(F')} \int_{F'} g d\omega \right)^{q'} \omega(F') \\
& = \sum_{F \in \mathcal{F}} \sum_{\substack{G \in \mathcal{G}: \\ \pi_{\mathcal{F}}(G)=F}} \sum_{\substack{F' \in ch_{\mathcal{F}}(F): \\ \pi_{\mathcal{G}}(F')=G}} \left( \frac{1}{\omega(F')} \int_{F'} g d\omega \right)^{q'} \omega(F') \\
& \leq \sum_{F \in \mathcal{F}} \sum_{\substack{G \in \mathcal{G}: \\ \pi_{\mathcal{F}}(G)=F}} \left( \frac{2}{\omega(G)} \int_G g d\omega \right)^{q'} \omega(G) \\
& \leq \sum_{G \in \mathcal{G}} \left( \frac{2}{\omega(G)} \int_G g d\omega \right)^{q'} \omega(G) \\
& \leq 2 \cdot 2^{q'} \|M_\omega g\|_{L^{q'}(d\omega)}^{q'} \leq 2 \cdot 2^{q'} q^{q'} \|g\|_{L^{q'}(d\omega)}^{q'}.
\end{aligned}$$

All together, we obtain

$$\sum_{Q \subset Q_0} K(Q) \int_Q f d\sigma \int_Q g d\omega \leq C \|\mathcal{W}_{K,\omega}^q[\sigma]^{1/q}\|_{L^r(d\sigma)} \|f\|_{L^p(d\sigma)} \|g\|_{L^{Q'}(d\omega)}.$$

This yields the sufficiency of Theorem 1.3.

**Proof of Theorem 1.3 (Necessity):** This fact was verified in [1, Theorem B (i)]. But, for reader's convenience the full proof is given here. We assume that the trace inequality (1.1) holds. Then, by Lemma 2.1, there holds

$$(2.4) \quad \sum_{Q \in \mathcal{D}} K(Q) \omega(Q) \int_Q f d\sigma \left( \frac{1}{\omega(Q)} \sum_{Q' \subset Q} K(Q') \omega(Q') \int_{Q'} f d\sigma \right)^{q-1} \leq C C_1^q \|f\|_{L^p(d\sigma)}^q,$$



where  $f \in L^p(d\sigma)$  is nonnegative. For  $g \geq 0$  we have

$$\begin{aligned}
& \int_{\mathbb{R}^n} g(x) \mathcal{W}_{K,\omega}^q[\sigma](x) d\sigma(x) \\
&= \sum_{Q \in \mathcal{D}} K(Q) \omega(Q) \int_Q g d\sigma \left( \frac{1}{\omega(Q)} \sum_{Q' \subset Q} K(Q') \omega(Q') \sigma(Q') \right)^{q-1} \\
&= \sum_{Q \in \mathcal{D}} K(Q) \omega(Q) \sigma(Q) \left( \frac{\int_Q g d\sigma}{\sigma(Q)} \right)^{1/q} \left( \frac{1}{\omega(Q)} \left( \frac{\int_Q g d\sigma}{\sigma(Q)} \right)^{1/q} \sum_{Q' \subset Q} K(Q') \omega(Q') \sigma(Q') \right)^{q-1} \\
&\leq \sum_{Q \in \mathcal{D}} K(Q) \omega(Q) \int_Q (M_\sigma g)^{1/q} d\sigma \left( \frac{1}{\omega(Q)} \sum_{Q' \subset Q} K(Q') \omega(Q') \int_{Q'} (M_\sigma g)^{1/q} d\sigma \right)^{q-1} \\
&\leq C C_1^q \| (M_\sigma g)^{1/q} \|_{L^p(d\sigma)}^q \\
&\leq C C_1^q \| g \|_{L^{p/q}(d\sigma)},
\end{aligned}$$

where we have used (2.4) and the  $L^{p/q}(d\sigma)$ -boundedness of  $M_\sigma$ . This implies by duality

$$\| \mathcal{W}_{K,\omega}^q[\sigma]^{1/q} \|_{L^r(d\sigma)} \leq C C_1 < \infty.$$

To verify

$$\| \mathcal{W}_{K,\sigma}^{p'}[\omega]^{1/p'} \|_{L^r(d\omega)} \leq C C_1 < \infty,$$

we merely use the dual inequality of (1.1).

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